M.Sc. – Third Semester/Second Year

Examination 2022
Subject: Mathematics
Paper: MMA-C301
Functional Analysis

Time: 3 Hrs. **Max. Marks: 70**
Min. Pass: 40%.

Section-A

Note: Attempt any five questions. All questions are having six marks.

Q.1 Is any Cauchy sequence in normed space convergent or not? To be Discussed its by one example.

Q.2 Define Banach Space and give an example.

Q.2 Let X be the vector space of all ordered pairs $x=(\xi_1,\xi_2)$, $y=(\eta_1,\eta_2)$, … of real numbers. Show that norm on X are defined by $\|x\|_4=(\xi_1^4+\xi_2^4)^{1/4}$.

Q.3 Can every metric on a vector space be obtained from a norm? It's Justify by a counterexample.

Q.4 Prove that a contraction T on a metric space X is a continuous mapping.

Q.5 Prove that the Schwarz inequality

$$|<x, y>| \leq ||x||\,.\,||y||,$$

where the equality sign holds if and only if {x, y} is a linearly dependent set.

Q.6 Define sub-linear functional and give one of the example.

Q.7 Give the definition of inner product space. Prove that the space C[a, b] is not an inner product space.

Q.8 Prove that Euclidean space R^3 is inner product space.

Q.9 Give one example which is norm space but not inner product space. Also, justify it.

Q.10 For any subset $M \neq \Phi$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

Section-B

Note: Attempt any four questions. All questions are having ten marks.

Q.1 Let T: $\mathfrak{D}(T) \to Y$ be a linear operator, where $\mathfrak{D}(T) \subset X$ and X, Y are normed spaces. Prove that T is continuous if and only if T is bounded.

Q.2 State and prove Riesz's lemma for norm space.

Q.3 State and prove open mapping theorem.

Q.4 Prove that a bounded linear operator T on a complex Hilbert space H is unitary if and only if is isometric and surjective.

Q.5 Prove that the space l^2 is an inner product, where l^2 is appear in own usual sense.

Q.6 Let $T \in B(X,X)$, where X is a Banach space and $B(X,X)$ be the bounded linear space. If $\|T\| < 1$, then prove that $(I-T)^{-1}$ exists as a bounded linear operator on the whole space X and $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$.

Q.7 Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint operators.

Q.8 If T: X→X satisfies d(Tx, Ty) < d(x, y) when x ≠ y and T has a fixed point, show that the fixed point is unique; here (X, d) is a metric space.
